数理解析研究所講究録 1464 巻 2006 年 29-39

On exterior Neumann problems with an asymptotically linear nonlinearity

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1 Introduction

We consider the following semilinear elliptic problem in an exterior domain with a Neumann boundary condition:

$$-\Delta u+u=f(u) \quad \text{in } \mathbb{R}^N\setminus\overline{\Omega}, \tag{1.1}$$

$$\frac{\partial u}{\partial\nu}=0 \quad \text{on } \partial\Omega,$$

where $\Omega\subset\mathbb{R}^N$ is an open bounded domain with $\partial\Omega\in C^1$, $N\geq 3$ and ν is an interior unit normal vector on $\partial\Omega$.

We are interested in the existence of a ground state solution of (1.1). More precisely, we define a functional $I_\Omega : H^1(\mathbb{R}^N\setminus\overline{\Omega})\mapsto\mathbb{R}$ by:

$$I_\Omega(u)=\frac{1}{2}\int_{\mathbb{R}^N\setminus\overline{\Omega}}|\nabla u|^2+u^2dx-\int_{\mathbb{R}^N\setminus\overline{\Omega}}F(u)dx$$

where $F(s)=\int_0^s f(t)dt$. A solution of (1.1) is called the ground state solution of (1.1) if it achieves $\inf\{I_\Omega(u); u\in H^1(\mathbb{R}^N\setminus\overline{\Omega})\setminus\{0\},\ I'_\Omega(u)=0\}$. In [5], Esteban established the existence of ground state solutions in the case $f(s)=|s|^{p-2}s$, $2<p<\frac{2N}{N-2}$.

Our first purpose is to obtain the existence of ground state solutions of (1.1) with an asymptotically linear nonlinearity. We assume
(f0) $f\in C^1(\mathbb{R}^+,\mathbb{R})$, $f(s)\equiv 0$ for all $s\leq 0$,
(f1) $\frac{f(s)}{s}\to 0$ as $s\to 0^+$, (f2) $\frac{f(s)}{s}\to a$ as $s\to\infty$, $1<a<\infty$,
Let $G(s)=\frac{1}{2}f(s)s-F(s)$. Then
(f3) (i) $G(s)\geq 0$ for all $s\geq 0$,
(ii) There exists $\delta_0\in(0,1)$ such that if $\frac{2F(s)}{s^2}\geq 1-\delta_0$, then $G(s)\geq\delta_0$.
Then we obtain the following result.

Theorem 1.1. *Let Ω be an open bounded domain with $\partial\Omega\in C^1$ and assume (f0)-(f3). Then problem (1.1) has a ground state solution.*

Our second purpose is to study a symmetry breaking phenomenon when Ω is a ball. In the case $f(s)=|s|^{p-2}s$, Esteban ([5]) also showed that ground state solutions of (1.1) are not radially symmetric. Moreover recently, Montefusco [11] showed that the non-radial ground state solution has an axial symmetry with respect to the line $r_P=0P$, where P is a maximum point.

Our question is that such a phenomenon occurs in the asymptotically linear nonlinearity case. The next theorem gives a positive answer to the question.

Theorem 1.2. *Let $\Omega = B_R(0) := \{x \in \mathbb{R}^N; |x| < R\}$ and assume (f0)-(f3). Then for every $R > 0$, the ground state solution of (1.1) is not radially symmetric.*

Finally we consider asymptotic profiles of ground state solutions of (1.1) when $\Omega = B_R(0)$. We denote χ_D by the characteristic function of a set $D \subset \mathbb{R}^N$.

Theorem 1.3. *Let $w_R(x)$ be a ground state solution of (1.1) with $\Omega = B_R(0)$. Then there exists $x_R \in \partial B_R(0)$ such that, passing to a subsequence,*

$$\|w_R - w(\cdot - x_R)\chi_{\mathbb{R}^N \setminus \overline{B_R(0)}}(\cdot)\|_{H^1(\mathbb{R}^N \setminus \overline{B_R(0)})} \to 0 \text{ as } R \to \infty,$$

where $w(x) \in H^1(\mathbb{R}^N)$ is a ground state solution to the problem:

$$-\Delta u + u = f(u) \text{ in } \mathbb{R}^N.$$

Recently asymptotically linear problems on $\mathbb{R}^N$ has been studied widely. Especially our assumptions on the nonlinearity $f(s)$ are based on those in [8]. The main difficulty of asymptotically linear problems is to obtain boundedness of Cerami sequences.

We also mention that to find a ground state solution in elliptic problems, it is usually assumed that $f(s)$ satisfies

$$s \mapsto \frac{f(s)}{s} \text{ is nondecreasing.} \tag{1.2}$$

Actually if (1.2) is satisfied, then the Nehari manifold:

$$N_\Omega = \{u \in H^1(\mathbb{R}^N \setminus \overline{\Omega}) \setminus \{0\}; I'_\Omega(u)u = 0\}$$

has nice properties. More precisely, a ground state solution $w(x)$ of (1.1) has the characterization:

$$I_\Omega(w) = \inf_{u \in N_\Omega} I_\Omega(u) = \inf_{u \in N_\Omega} \max_{t>0} I_\Omega(tu). \tag{1.3}$$

However in this paper, we don't require (1.2). We will find a Mountain Pass solution and after that, we prove the existence of a ground state solution.

To prove Theorem 1.2, we will compare the ground state energy level for (1.1) with the radially symmetric one and obtain a strict gap between them. If $f(s)$ satisfies

$$0 < \mu f(s) \leq f'(s)s \text{ for all } s > 0 \tag{1.4}$$

for some $\mu > 1$, then we can see that for every $u \in H^1(\mathbb{R}^N \setminus \overline{B_R(0)}) \setminus \{0\}$, there exists a unique $k > 0$ such that $ku \in N_{B_R(0)}$. This fact and the chacterization (1.3) are useful to show the energy gap. However (1.4) implies $f(s)s^{-q}$ is nondecreasing for all $1 \leq q \leq \mu$. This means that $f(s)$ never satisfy (1.4) and (f2) at the same time. Furthermore we can't use the characterization (1.3) because we don't assume (1.2). Making use of the Pohozaev type identity as a main tool, we will prove Theorem 1.2 (see also Remark 4.2 below).

Finally exterior Neumann problems which concern with multiplicity results or multipeak solutions have been studied in [2], [4], [12], [13]. Especially our asymptotic profile of Theorem 1.3 corresponds to that of the singularly perturbed problem with a fixed radius:

$$-\epsilon^2 \Delta u(x) + u(x) = f(u(x)), \ x \in \mathbb{R}^N \setminus B_1(0).$$

2 Some results for problems in $\mathbb{R}^N$

We consider the problem:

$$-\Delta u = f(u) - u =: h(u) \text{ in } \mathbb{R}^N. \tag{2.1}$$

In this section, we recall some known results for (2.1). Although results below are obtained under weaker assumptions on the nonlinearity, we do not provide precise statements here.

Proposition 2.1. *([1]) Assume (f0)-(f2), then (2.1) has a positive ground state solution $w_0(x) \in C^2(\mathbb{R}^N)$ and it satisfies*
(i) $w_0(x)$ is radially symmetric with respect to the origin (up to translation).
(ii) $|D^\alpha w_0(x)| \le Ce^{-\delta|x|}$ $x \in \mathbb{R}^N$ for some C, $\delta > 0$ and for $0 \le |\alpha| \le 2$.

Proposition 2.2. *([6], [9]) Assume (f0)-(f2), then every positive solution of (2.1) are radially symmetric with respect to the origin (up to translation) and satisfy (ii) in Proposition 2.1.*

Proposition 2.3. *([1]) Assume (f0)-(f2). Let $u(x)$ be a solution of (2.1). Then $u(x)$ satsifies the Pohozaev type identity:*

$$\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx = N\int_{\mathbb{R}^N} H(u)dx, \tag{2.2}$$

where $H(s) = \int_0^s h(t)dt$.

We define a functional $I_0 : H^1(\mathbb{R}^N) \mapsto \mathbb{R}$ by

$$I_0(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 + u^2 dx - \int_{\mathbb{R}^N} F(u)dx.$$

We denote m_0 by a ground state energy level, i.e.

$$m_0 = \inf\{I_0(u); u \in H^1(\mathbb{R}^N) \setminus \{0\}, I_0'(u) = 0\}.$$

Finally we define a Mountain Pass value of I_0.

$$c_0 := \inf_{\gamma\in\Gamma_0} \max_{t\in[0,1]} I_0(\gamma(t)),$$

$$\Gamma_0 := \{\gamma \in C([0,1], H^1(\mathbb{R}^N)); I_0(0) = 0, I_0(\gamma(1)) < 0\}.$$

Proposition 2.4. *([7]) Assume (f0)-(f2). Then $c_0 = m_0$, that is, the Mountain Pass value of I_0 is the ground state energy level.*

3 Proof of Theorem 1.1

The purpose of this section is to establish the existence of a ground state solution of (1.1). First we prove the existence of a Mountain Pass solution.

Lemma 3.1. *Assume (f0)-(f2). Then*
(i) $I_\Omega(u) = \frac{1}{2}\|u\|^2_{H^1(\mathbb{R}^N\setminus\overline{\Omega})} + o(\|u\|^2_{H^1(\mathbb{R}^N\setminus\overline{\Omega})})$.
(ii) There exists $v \in H^1(\mathbb{R}^N \setminus \overline{\Omega}) \setminus \{0\}$ such that $I_\Omega(v) < 0$.

The proof of Lemma 2.1 (i) is standard. The second part is not trivial because the nonlinearity is asymptotically linear. See Lemma 3.3 below or [7] for the proof.

By Lemma 2.1, we can define the Mountain Pass value for I_Ω:

$$c_\Omega := \inf_{\gamma\in\Gamma_\Omega}\max_{t\in[0,1]} I_\Omega(\gamma(t)),$$

$$\Gamma_\Omega = \{\gamma \in C([0,1], H^1(\mathbb{R}^N \setminus \overline{\Omega})); \gamma(0) = 0, I_\Omega(\gamma(1)) < 0\}.$$

Lemma 3.2. *Assume (f0)-(f3). Let* $\{u_n\} \subset H^1(\mathbb{R}^N \setminus \overline{\Omega})$ *be a sequence such that*

$$I_\Omega(u_n) \to c_\Omega,\ I'_\Omega(u_n)(1 + \|u_n\|) \to 0 \text{ as } n \to \infty.$$

Then the sequence $\{u_n\}$ *is bounded.*

In the proof of Lemma 3.2, assumption (f3) plays an important role.

Sketch of the proof of Lemma 3.2. We suppose by contradiction that

$$\|u_n\|_{H^1(\mathbb{R}^N\setminus\overline{\Omega})} \to \infty \text{ as } n \to \infty$$

and put

$$v_n(x) := \frac{u_n(x)}{\|u_n\|_{H^1(\mathbb{R}^N\setminus\overline{\Omega})}}.$$

Then by concentration compactness principle [10], one of the following statements holds.

1: (Vanishing) For all $r > 0$,

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_r(y)\setminus\overline{\Omega}} v_n^2 dx = 0. \tag{3.1}$$

2: (Non-vanishing) There exist $\alpha > 0$, $r_0 \in (0,\infty)$ and $\{y_n\} \subset \mathbb{R}^N \setminus \overline{\Omega}$ such that

$$\lim_{n\to\infty}\int_{B_{r_0}(y_n)\setminus\overline{\Omega}} v_n^2 dx \geq \alpha. \tag{3.2}$$

We show that both of them derive contradictions.

Step 1:(3.1) is impossible.

Here we use assumption (f3). We assume (3.1). We define

$$\Omega_n := \{x \in \mathbb{R}^N \setminus \overline{\Omega}; \frac{F(u_n(x))}{u_n(x)^2} \leq \frac{1}{2}(1-\delta_0)\}$$

where δ_0 is the constant defined in (f3) (ii). Making use of assumption (3.1), we obtain

$$\limsup_{n\to\infty} |\mathbb{R}^N \setminus (\overline{\Omega} \cup \Omega_n)| = \infty.$$

Then from (f3), we have

$$\int_{\mathbb{R}^N\setminus\overline{\Omega}} G(u_n)dx \geq \int_{\mathbb{R}^N\setminus(\overline{\Omega}\cup\Omega_n)} G(u_n)dx \geq \delta_0|\mathbb{R}^N \setminus (\overline{\Omega}\cup\Omega_n)|$$

and hence

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N\setminus\overline{\Omega}} G(u_n)dx = \infty.$$

On the other hand, we also have

$$\int_{\mathbb{R}^N\setminus\overline{\Omega}} G(u_n)dx = I_\Omega(u_n) - \frac{1}{2}I'_\Omega(u_n)u_n \to c < \infty.$$

This is a contradiction.

Step 2: (3.2) is impossible.
We assume (3.2) and $\{y_n\}$ is bounded. Since $\{v_n\}$ is bounded, we may assume that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N \setminus \overline{\Omega})$. Then we can show that $v(x)$ should be an eigenfunction of $-\Delta$ on $L^2(\mathbb{R}^N \setminus \overline{\Omega})$ corresponding to the eigenvalue $a-1$. However this is a contradiction because $-\Delta$ on $L^2(\mathbb{R}^N \setminus \overline{\Omega})$ has no eigenvalues (see [14]).

Finally we assume (3.2) and $\{y_n\}$ is unbounded. Since $\partial\Omega \in C^1$, there exists an extension operator $E : H^1(\mathbb{R}^N\setminus\overline{\Omega}) \mapsto H^1(\mathbb{R}^N)$. We put $\tilde{v}_n(x) := Ev_n(x+y_n)$. Then we can show that a weak limit of $\tilde{v}_n(x)$ should be an eigenfunction of $-\Delta$ on $L^2(\mathbb{R}^N)$, which is a contradiction. □

Next we estimate a Mountain Pass value of I_Ω. It is rather standard to show that $c_\Omega \le m_0$. We show that this inequality is strict.

Lemma 3.3. *Assume (f0)-(f2). Then* $c_\Omega < m_0$.

Here we give an outline of the proof.

Proof of Lemma 3.3. By definition of c_Ω, it is sufficient to show that there exists a path $\gamma_0 \in \Gamma_\Omega$ such that

$$\max_{t\in[0,1]} I_\Omega(\gamma_0(t)) < m_0.$$

We construct such a path γ_0 as follows.

Let $w_0(x)$ be a ground state solution of (2.1). First we show there exists $t_0 > 1$ independent of $z \in \mathbb{R}^N$ such that

$$I_\Omega(w_0(\frac{x-z}{t_0})\chi_{\mathbb{R}^N\setminus\overline{\Omega}}(x)) < 0. \tag{3.3}$$

Indeed by Proposition 2.3, we obtain

$$I_\Omega(w_0(\frac{x-z}{t})\chi_{\mathbb{R}^N\setminus\overline{\Omega}}(x))$$

$$\le (\frac{t^{N-2}}{2} - \frac{N-2}{2N}t^N)\|\nabla w_0\|^2_{L^2(\mathbb{R}^N)} + \sup_{x\in\mathbb{R}^N} |F(w_0(x))||\overline{\Omega}|.$$

Thus we can choose $t_0 > 1$ so that (3.3) holds.

Next let $0 < \delta < m_0$ be given. Then we can easily show that there exists $t_1 > 0$ such that

$$I_\Omega(w_0(\frac{x-z}{t})\chi_{\mathbb{R}^N\setminus\overline{\Omega}}(x)) < \delta \tag{3.4}$$

for all $0 < t < t_1$ and $z \in \mathbb{R}^N$.

Finally we show that

$$\max_{t\in[t_1,t_0]} I_\Omega(w_0(\frac{x-z}{t})\chi_{\mathbb{R}^N\setminus\overline{\Omega}}(x)) < m_0 \tag{3.5}$$

for some $z_0 \in \mathbb{R}^N \setminus \overline{\Omega}$. In fact, we can estimate as follows:

$$I_\Omega(w_0(\frac{x-z}{t})\chi_{\mathbb{R}^N\setminus\overline{\Omega}}(x))$$

$$\leq m_0 - \frac{t^N}{2}\int_{\frac{1}{t}(\overline{\Omega}+z)} w_0^2 dx + t^N \int_{\frac{1}{t}(\overline{\Omega}+z)} |F(w_0)| dx.$$

Then by the decay property of w_0 (Prop. 2.1 (ii)), we obtain

$$\max_{t\in[t_1,t_0]}\{-\frac{1}{2}\int_{\frac{1}{t}(\overline{\Omega}+z_0)} w_0^2 dx + \int_{\frac{1}{t}(\overline{\Omega}+z_0)} |F(w_0)| dx\} < 0$$

for some $z_0 \in \mathbb{R}^N$.

Now we define

$$\gamma_0(t) := \begin{cases} w_0(\frac{x-z_0}{tt_0}) & 0 < t \leq 1, \\ 0 & t = 0. \end{cases}$$

Then from (3.3)-(3.5), $\gamma_0(t) \in \Gamma_\Omega$ and $\max_{t\in[0,1]} I_\Omega(\gamma_0(t)) < m_0$. □

Now by Lemma 3.1-3.3, we can show there exists $u_0 \in H^1(\mathbb{R}^N \setminus \overline{\Omega})$ such that

$$I'_\Omega(u_0) = 0 \text{ and } I_\Omega(u_0) = c_\Omega.$$

Since $c_\Omega > 0$, it follows $u_0 \neq 0$. Especially,

$$\{u \in H^1(\mathbb{R}^N \setminus \overline{\Omega}) \setminus \{0\}; I'_\Omega(u) = 0\} \neq \emptyset.$$

Proposition 3.4. *Assume (f0)-(f3). Then (1.1) has a ground state solution.*

Proof. First we define the ground state energy level for (1.1) by

$$m_\Omega := \inf\{I_\Omega(u); u \in H^1(\mathbb{R}^N \setminus \overline{\Omega}) \setminus \{0\},\ I'_\Omega(u) = 0\}.$$

From (f3) (i), for any non-trivial critical point u of I_Ω, we have

$$I_\Omega(u) = I_\Omega(u) - \frac{1}{2}I'_\Omega(u)u = \int_{\mathbb{R}^N\setminus\overline{\Omega}} G(u) dx \geq 0.$$

Thus $m_\Omega \geq 0$. On the other hand, it is trivial that $m_\Omega \leq c_\Omega$.

Now let $\{w_n\} \subset \{u \in H^1(\mathbb{R}^N \setminus \overline{\Omega}) \setminus \{0\}; I'_\Omega(u) = 0\}$ be a sequence such that $I_\Omega(w_n) \to m_\Omega \in [0, c_\Omega]$. Then $\{w_n\}$ is bounded and

$$\liminf_{n\to\infty} \|w_n\|_{H^1(\mathbb{R}^N\setminus\overline{\Omega})} \geq \rho_0$$

for some $\rho_0 > 0$. Thus we may assume that $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N \setminus \overline{\Omega})$. Then we obtain $I_\Omega(w_n) \to I_\Omega(w)$ and $\|w\| \geq \rho_0$. Thus we have $I_\Omega(w) = m_\Omega$ and $w \neq 0$, that is, $w(x)$ is a ground state solution. □

Theorem 1.1 is a consequence of Proposition 3.4. In the proof of Proposition 3.4, we know that $m_\Omega \geq 0$. We can show that $m_\Omega > 0$.

Finally we prepare a Pohozaev type identity which plays an important role in the next section.

Proposition 3.5. *Assume (f0)-(f1) and $f(s)$ has a sub-critical growth at infinity. Let $u(x)$ be a solution of (1.1). Then $u(x)$ satisfies the following Pohozaev type identity:*

$$\frac{N-2}{2}\int_{\mathbb{R}^N\setminus\overline{\Omega}}|\nabla u|^2dx = N\int_{\mathbb{R}^N\setminus\overline{\Omega}}H(u)dx - \int_{\partial\Omega}H(u)x\cdot\nu d\sigma,$$

where ν is an interior unit normal vector on $\partial\Omega$.

4 Proof of Theorem 1.2

Hereafter we consider problem (1.1) with $\Omega = B_R(0)$:

$$-\Delta u + u = f(u) \text{ in } \mathbb{R}^N\setminus\overline{B_R(0)}, \tag{4.1}$$
$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial B_R(0).$$

By Theorem 1.1, (4.1) has a ground state solution $w_R(x)$ for every $R > 0$. For simplicity, we write $m_{B_R(0)} = m_R$, $c_{B_R(0)} = c_R$, $I_{B_R(0)} = I_R$.

We define $H_R^* := \{u \in H^1(\mathbb{R}^N\setminus\overline{B_R(0)}); u(x) = u(|x|)\}$,

$$m_R^* = \inf\{I_R(u); u \in H_R^*\setminus\{0\},\ I_R'(u) = 0\}.$$

Then we can show that m_R^* is achieved.

Now we turn to the proof of Theorem 1.2. By definitions, it is trivial that $m_R \leq m_R^*$. We show that this inequality is strict for every $R > 0$. By Lemma 3.3, we already know that $m_R < m_0$ for every $R > 0$. Thus we have only to show that $m_0 \leq m_R^*$. Indeed, we obtain the following estimate.

Proposition 4.1. *For every $R > 0$, $m_0 < m_R^*$.*

Proof. Now by Proposition 2.4, we know $c_0 = m_0$. Thus it is sufficient to show that there exists a path $\gamma(t) \in \Gamma_0$ such that $\max_{t\in[0,1]} I_0(\gamma(t)) < m_R^*$. The proof consists of three steps.

Step 1: Formulation of m_R^*.

Let w_R^* be a radial ground state solution of (4.1). Then by Proposition 3.5, w_R^* satisfies

$$\begin{aligned}\frac{N-2}{2}\int_{\mathbb{R}^N\setminus\overline{B_R(0)}}|\nabla w_R^*|^2dx &= N\int_{\mathbb{R}^N\setminus\overline{B_R(0)}}H(w_R^*)dx + R\int_{\partial B_R(0)}H(w_R^*)d\sigma\\ &= N\int_{\mathbb{R}^N\setminus\overline{B_R(0)}}H(w_R^*)dx + R^N|S^{n-1}|H(w_R^*(R)).\end{aligned}$$

Then we obtain

$$m_R^* = \frac{1}{2}\int_{\mathbb{R}^N\setminus\overline{B_R(0)}}|\nabla w_R^*|^2dx - \int_{\mathbb{R}^N\setminus\overline{B_R(0)}}H(w_R^*)dx$$

$$= \frac{1}{N}\int_{\mathbb{R}^N\setminus\overline{B_R(0)}} |\nabla w_R^*|^2 dx + \frac{1}{N}R^N|S^{N-1}|H(w_R^*(R)).$$

On the other hand, since $w_R^*(x)$ is radially symmetric, we may assume that $w_R^*(r)$ satisfies the following ODE:

$$-(w_R^*)''(r) - \frac{N-1}{r}(w_R^*)'(r) = h(w_R^*(r)),\ R < r < \infty,\ (w_R^*)'(R) = 0.$$

Multiplying $(w_R^*)'$ in both sides and integrating over (R,∞), we obtain

$$-\frac{1}{2}\int_R^\infty \frac{d}{dr}((w_R^*)')^2 dr - (N-1)\int_R^\infty \frac{((w_R^*)')^2}{r}dr = \int_R^\infty (H(w_R^*))'dr.$$

Thus we have

$$0 < (N-1)\int_R^\infty \frac{((w_R^*)')^2}{r}dr = H(w_R^*(R)).$$

Now for simplicity, we write

$$A = \int_{\mathbb{R}^N\setminus\overline{B_R(0)}} |\nabla w_R^*|^2 dx,\ B = R^N|S^{N-1}|H(w_R^*(R)).$$

Then $A,\ B > 0$ and we have $m_R^* = \frac{1}{N}(A+B)$.

Step 2: Construction of a path.

Now we define

$$\tilde{w}_R(x) = \begin{cases} w_R^*(x) & |x| > R, \\ w_R^*(R) & |x| \le R. \end{cases}$$

Then $\tilde{w}_R(x) \in H^1(\mathbb{R}^N)$ and

$$I_0(\tilde{w}_R(\frac{x}{t})) = (\frac{t^{N-2}}{2} - \frac{N-2}{2N}t^N)\int_{\mathbb{R}^N\setminus\overline{B_R(0)}} |\nabla w_R^*|^2 dx$$

$$+\frac{R}{N}t^N\int_{\partial B_R(0)} H(w_R^*)d\sigma - t^N\int_{\overline{B_R(0)}} H(w_R^*(R))dx$$

$$= (\frac{t^{N-2}}{2} - \frac{N-2}{2N}t^N)A + (\frac{1}{N}-1)t^N B.$$

Since $A,\ B > 0$, there exists $t_0 > 1$ such that $I_0(\tilde{w}_R(\frac{x}{t_0})) < 0$. Putting

$$\gamma_R(t) := \begin{cases} \tilde{w}_R(\frac{x}{tt_0}) & 0 < t \le 1, \\ 0 & t = 0, \end{cases}$$

then $\gamma_R(t) \in \Gamma_0$.

Step 3: Conclusion.

Now we have

$$I_0(\gamma_R(t)) = (\frac{(tt_0)^{N-2}}{2} - \frac{N-2}{2N}(tt_0)^N)A + (\frac{1}{N}-1)(tt_0)^N B =: C(t).$$

Then for $t > 0$, $C'(t) = 0$ if and only if t satisfies

$$1 = (tt_0)^2(1 + \frac{2(N-1)}{N-2}\frac{B}{A}).$$

We put
$$t_1 := \frac{1}{t_0}(1 + \frac{2(N-1)}{N-2}\frac{B}{A})^{-\frac{1}{2}}.$$
Since A, $B > 0$, we have $t_1 t_0 < 1$. Moreover $C(t) \le C(t_1)$ for all $t \in [0,1]$. Thus we get
$$\begin{aligned} I_0(\gamma_R(t)) &\le C(t_1) \\ &= (t_1 t_0)^{N-2}(\frac{A}{2} - (t_1 t_0)^2(\frac{(N-2)A + 2(N-1)B}{2N})) \\ &= \frac{1}{N}(t_1 t_0)^{N-2} A < \frac{1}{N} A < \frac{1}{N}(A+B) = m_R^*. \end{aligned}$$
Thus we obtain
$$\max_{t \in [0,1]} I_0(\gamma_R(t)) < m_R^*$$
and hence $m_0 < m_R^*$. □

Remark 4.2. *In the case* $f(s) = |s|^{p-2}s$, $2 < p < \frac{2N}{N-2}$, *Esteban [5] showed that*
$$R \mapsto m_R^* \text{ is increasing and } \lim_{R \to 0^+} m_R^* = m_0.$$
Same conclusions hold true under assumption (1.4), i.e.
$$0 < \mu f(s) \le f'(s)s \text{ for all } s > 0$$
for some $\mu > 1$ *(see [3]). In their proofs, they used nice characterizations of* m_R^* *(like (1.3) in section 1). In our proof, the key is the Pohozaev type identity, which is applicable to general nonlinearities. Especially in the proof of Proposition 4.1, we don't require that* $f(s)$ *is asymptotically linear.*

Although we don't know whether such a monotonicity of m_R^* *does follow or not in our situation, we can obtain the followings.*

Corollary 4.3. *(i) Let* $R' > 0$ *be given. Then there exists* $0 < R_0 < R'$ *such that* $m_R^* < m_{R'}^*$ *for all* $R \in (0, R_0)$. *(ii)* $\lim_{R \to 0^+} m_R^* = m_0$.

Proof of Theorem 1.2. Now by Lemma 3.3 and Proposition 4.1, we have
$$m_R \le c_R < m_0 < m_R^* \le c_R^*.$$
This inequality implies that the ground state solution of (4.1) is not radially symmetric. □

As a corollary, we obtain the following result.

Corollary 4.4. *Assume (f0)-(f3). Then problem (4.1) has at least two positive solutions; one is radially symmetric and the other is non-radial.*

5 Proof of Theorem 1.3

In this section, we give a sketch of the proof of Theorem 1.3.

Let $w_R(x)$ be a ground state solution of (4.1). Then we have the following lemma.

Lemma 5.1. *There exists $C > 0$ independent of large R such that*

$$\|w_R\|_{H^1(\mathbb{R}^N \setminus \overline{B_R(0)})} \leq C.$$

To complete the proof of Theorem 1.3, we show a limiting behavior of m_R as $R \to \infty$. More precisely, we will show that $\lim_{R\to\infty} m_R = \frac{1}{2}m_0$. The most difficult part of the proof of Theorem 1.3 is that we can't prove $\lim_{R\to\infty} m_R = \frac{1}{2}m_0$ directly. First we obtain the following estimates.

Lemma 5.2. *(i) There exists $C > 0$ such that $m_R \geq C$ for sufficiently large $R > 0$.*

(ii) $\limsup_{R\to\infty} m_R \leq \frac{1}{2}m_0$.

Proposition 5.3. $\lim_{R\to\infty} m_R = \frac{1}{2}m_0$. *Moreover let $w_R(x)$ be a ground state solution of (4.1). Then there exists $x_R \in \partial B_R(0)$ such that*

$$\|w_R - w(\cdot - x_R)\chi_{\mathbb{R}^N \setminus \overline{B_R(0)}}(\cdot)\|_{H^1(\mathbb{R}^N \setminus \overline{B_R(0)})} \to 0 \text{ as } R \to \infty,$$

where $w(x)$ is a ground state solution of (2.1).

This proposition completes the proof of Theorem 1.3. Here we just give an outline of the proof of Proposition 5.3 because the proof is rather complicated.

By Lemma 5.1 and 5.2 (i), there exists $y_R \in \mathbb{R}^N \setminus \overline{B_R(0)}$ such that

$$w_R(x) - w(x - y_R) \rightharpoonup 0 \text{ in } H^1(\mathbb{R}^N \setminus \overline{B_R(0)}) \text{ as } R \to \infty$$

where $w(x)$ is a ground state solution of (2.1). Then we have $dist(y_R, \partial B_R(0)) \leq C$ for some C independent of large R. We suppose by contradiction that $w_R(x) - w(x - y_R)$ does not converge to zero in $H^1(\mathbb{R}^N \setminus \overline{B_R(0)})$. Then we can show that $\liminf_{R\to\infty} m_R \geq m_0$, which contradicts to Lemma 5.2 (ii). Finally by the property of $w(x)$, we can complete the proof of Proposition 5.3.

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